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ANALYSIS OF TRANSMUTED MAXWELL DISTRIBUTION

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Maxwell distribution is recognized as a life time model in Statistics literature. In this paper, we develop transmuted Maxwell distribution using the quadratic rank transmutation map studied by Shaw and Buckley (2009). We discuss various mathematical properties of this distribution including some reliability measures. We obtain Maximum likelihood and Bayes estimators of parameters and provide confidence intervals and Bayesian credible intervals for the same. We perform simulation study and analyse a real data set for numerical illustrations.

Keywords: Transmuted Maxwell distribution, hazard rate, maximum likelihood estimator, asymptotic confidence intervals, Bayes estimator, credible intervals.

Introduction

Maxwell distribution, as a lifetime model, has been extensively studied in literature. Tyagi and Bhattacharya (1989a) considered minimum variance unbiased estimators of mean life and reliability function whereas Tyagi and Bhattacharya (1989b) discussed the Bayesian estimation for this distribution. Bekker and Roux (2005) derived empirical Bayes estimators of the rats parameter and hazard function. Krishna and Malik (2012) obtained maximum likelihood (*ML*) and Bayes estimators of the parameter and reliability function under progressive censoring scheme. Tomer and Panwar (2015) considered *ML* and Bayesian estimation of the parameter of this distribution under type-I progressive hybrid censoring schemes.

The quadratic rank transmutation map [see Shaw and Buckley (2009)] is one of the several approaches that are advocated in literature to generalize any distribution. Aryal and Tsokos (2009) studied the transmuted extreme value distribution and suggested its applications in the areas like climatology and hydrology. Merovci (2016) and Aryal and Tsokos (2011), respectively discussed the properties and parameter estimation of transmuted Rayleigh and Weibull distributions. Khan and King (2013) considered three parameter kumaraswamy distribution and obtianed estimators of various parametric functions. In this paper, we present transmuted Maxwell(*TMW*) distribution using the quadratic rank transmutation map.

A random variable X is said to have transmuted distribution if its cumulative distribution function (*cdf)* is given by

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \quad |\lambda| \le 1 \qquad (1)$$

where G(x) is the *cdf* of the base distribution. We observe that for $\lambda = 0$, we get the cdf of base distribution.

The rest of paper is organised as follows. In Section 2, we define the *TMW* distribution and obtain its mgf. In Section 3, we discuss various properties of *TMW* along with its reliability and hazard rate functions. In Section 4, we obtain the ML estimate and asymptotic confidence intervals(*ACIs*) for parameters. We provide procedures to evaluate Bayes estimators of parameters, Bayesian credible interval and highest posterior density intervals in Section 5. We perform simulation study in Section 6 and finally analyse a real data in Section 7.

2. Transmuted Maxwell distribution

It a random variable X follows Maxwell distribution with scale parameter θ then its *pdf* is given by

$$g(x;\theta) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{\frac{3}{2}}} x^2 e^{-\frac{x^2}{\theta}}; \quad 0 \le x < \infty, \quad \theta > 0,. \qquad (2)$$

and the *cdf* of X is given by

$$G(x;\theta) = 1 - \frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right), \qquad (3)$$

Now using (1) and (3), we get the *cdf* of TMW as follows

$$F(x,\theta,\lambda) = \left(1 - \frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right)\left(1 + \frac{2\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right) \qquad (4)$$

Hence the *pdf* of TMW with parameters θ and λ comes out to be

$$f(x,\theta,\lambda) = \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{\frac{3}{2}}} e^{-\frac{x^2}{\theta}} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right). \qquad (5)$$

Some possible shapes of *pdf* of *TMW* for different values of parameters θ and λ are shown in Figure 1.

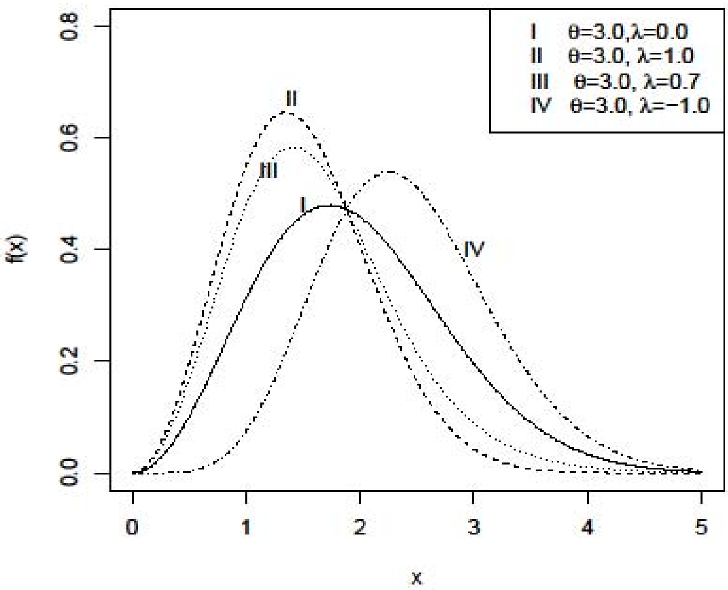

Figure1: The pdf's of various transmuted Maxwell distribution

2.1 Moment Generating Function

If X follows, *TMW* astin bolt (5), then its the moment generating function(*MGF*) is given by

$$M_X(t) = \mathrm{E}(e^{tX})$$

$$= \int_0^\infty \exp(tx) f(x,\theta,\lambda) dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x,\theta,\lambda) dx$$

$$= \sum_{i=1}^{\infty} \frac{t^i}{i!} E\left(X^i\right), \qquad (6\,)$$

For on integer r, we have

$$E(X^r) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{\frac{3}{2}}} \int_0^\infty x^{r+2}\, e^{\frac{-x^2}{\theta}} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right) dx$$

$$= \frac{2}{\sqrt{\pi}} \theta^{\frac{r}{2}} \left[(1-\lambda) \int_0^\infty \mathrm{t}^{\frac{r+1}{2}}\, \mathrm{e}^{-\mathrm{t}}\, \mathrm{dt} + \frac{4\lambda}{\sqrt{\pi}} \int_0^\infty \mathrm{t}^{\frac{r+1}{2}}\, \mathrm{e}^{-\mathrm{t}}\, \Gamma_{\frac{3}{2}}(\mathrm{t}) \mathrm{dt}\right],$$

which on using a result from Gradshteyn and Ryzhik (1980), pp 663,§6.455), comes out to be

$$E(X^r) = \frac{2\theta^{\frac{r}{2}}}{\sqrt{\pi}}\left[(1-\lambda)\Gamma\left(\frac{3+r}{2}\right) + \frac{4\lambda}{\sqrt{\pi}}\frac{\Gamma(3+\frac{r}{2})}{\left(\frac{3+r}{2}\right)2^{3+\frac{r}{2}}}\ {}_2F_1\left(1,3+\frac{r}{2};\frac{5+r}{2};\frac{1}{2}\right)\right] \quad (7)$$

where ${}_2F_1$ is Gauss Hypergeometric function.

Using (6) and (7), we get the mgf of TMW distribution as follows.

$$M_X(t) = \frac{2}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{t^i}{i!}\theta^{\frac{i}{2}}\left[(1-\lambda)\Gamma\left(\frac{3+i}{2}\right) + \frac{4\lambda}{\sqrt{\pi}}\frac{\Gamma\left(3+\frac{i}{2}\right)}{\left(\frac{3+i}{2}\right)2^{3+\frac{i}{2}}}\ {}_2F_1\left(1,3+\frac{i}{2};\frac{5+i}{2};\frac{1}{2}\right)\right], \quad (8)$$

In perticular, we have

$$E(X) = 2\sqrt{\frac{\theta}{\pi}}\left[1-\lambda+\frac{\lambda}{\sqrt{\pi}}\frac{\Gamma\left(\frac{7}{2}\right)}{2^{\frac{5}{2}}}\ {}_2F_1\left(1,\frac{7}{2};3;\frac{1}{2}\right)\right]$$

and

$$E(X^2) = \frac{2\theta}{\sqrt{\pi}}\left[(1-\lambda)\Gamma\left(\frac{5}{2}\right) + \frac{3\lambda}{5\sqrt{\pi}}\ {}_2F_1\left(1,4;\frac{7}{2};\frac{1}{2}\right)\right].$$

3. Reliability Analysis

The reliability function of a *TMW,* at any specified time t, is

$$R(t) = P(\mathrm{X} > t)$$

$$= \frac{2}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\left[1-\lambda+\frac{2\lambda}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right]. \quad (9)$$

The hazard rate function for this distribution is

$$h(t) = \frac{f(t)}{R(t)}$$

$$= \frac{2t^2}{\theta^{\frac{3}{2}}}\frac{\exp(-\frac{t^2}{\theta})}{\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)}\left[\frac{1-\lambda+\frac{4\lambda}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)}{1-\lambda+\frac{2\lambda}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)}\right]. \quad (10)$$

In Figure 2, the hazard function (10) is ploted for different Values of θ and λ.

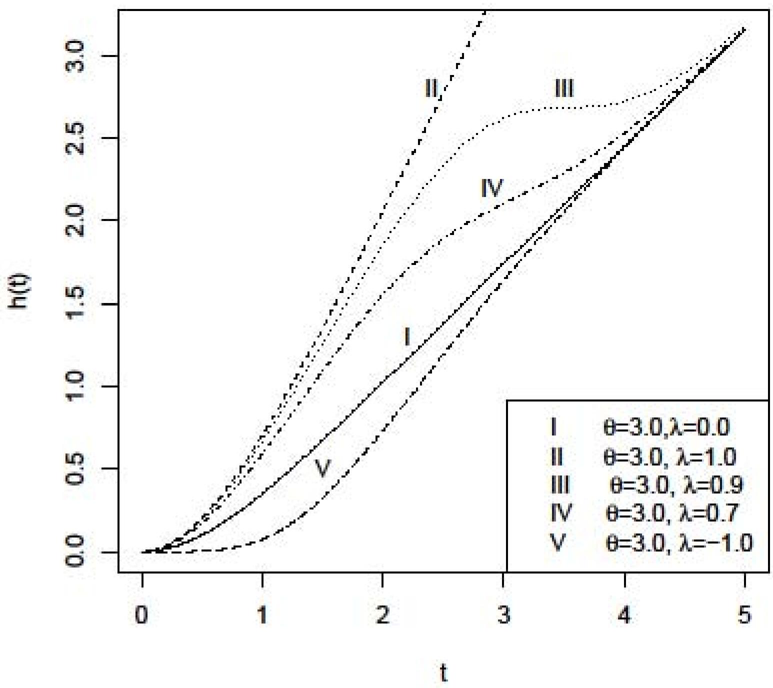

Figure 2: The hazard function of *TMW* for various parameteric values.

3.1 **Mean residual life**

The mean residual life function (*MRL*) at a given time t measures the expected remaining lifetime of an individual of age t. It is denoted by m(x). For a continuous distribution with *pdf* f(x) and *cdf* F(x), the mean residual life function is defined as

$$m(x) = E(X - x|X > x) = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(t)]dt.$$

For *TMW,* the mean residual life function comes out to be

$$m(x) = \frac{1}{\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\left[1-\lambda+\frac{2\lambda}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right]} \int_x^\infty \Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\left(1 - \lambda + \frac{2\lambda}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right)dt.$$

3.2 Order Statistics

Let $X_1 \leq X_2 \leq \ldots \leq X_n$ denote the order statistics of a random sample from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the *pdf* of j^{th} order statistic X_j is given by

$$f_{X_j}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)(F_X(x))^{j-1}(1 - F_X(x))^{n-j}, j = 1, \ldots, n. \quad (11)$$

The *pdf* of the j^{th} order statistic when X follows TMW is given by

$$f_{X_j}(x) = \frac{n!}{(j-1)!\,(n-j)!} \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{\frac{3}{2}}} e^{-\frac{x^2}{\theta}} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right) \left[\left(1 - \frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right)\right.$$

$$\left.\left(1 + \frac{2\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right)\right]^{j-1} \left[\left(\frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right)\left(1 - \lambda + \frac{2\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right)\right]^{n-j}. \quad (12)$$

In particular, the *pdf* of the largest order statistic X_n comes out to be

$$f_{X_n}(x) = n \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{\frac{3}{2}}} e^{-\frac{x^2}{\theta}} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right) \left[\left(1 - \frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right)\left(1 + \frac{2\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right)\right]^{n-1}$$

and, the *pdf* of the smallest order statistic X_1 be comes

$$f_{X_1}(x) = n \frac{4}{\sqrt{\pi}} \frac{x^2}{\theta^{\frac{3}{2}}} e^{-\frac{x^2}{\theta}} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta}\right)\right) \left[\left(\frac{2}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right)\left(1 - \lambda + \frac{2\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{t^2}{\theta}\right)\right)\right]^{n-1}.$$

3.3 Stochastic orderings

Stochastic ordering of positive continuous random variables is an important tool to judge their comparative behavior. There are different types of stochastic orderings which are useful in ordering random variables. Here we consider four different stochastic orders, namely, stochastic order, the hazard rate, the mean residual life, and the likelihood ratio order for two independent *TMW* random variables under a restricted parameter space.

A random variable X is said to be smaller than a random variable Y in the

1. stochastic(st) order $(X \leq_{st} Y)$ if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x;
2. hazard rate(hr) order $(X \leq_{hr} Y)$ if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in x;
3. mean residual life(mrl) order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x;
4. likelihood ratio(lr) order $(X \leq_{lr} Y)$ if $f_Y(x)/f_X(x)$ is increasing in x.

The hazard rate ordering is also known as uniform stochastic ordering in the literature.

The *TMW* distribution is ordered with respect to the strongest i.e. likelihood ratio ordering as shown in the following theorem.

Theorem 1. If X and Y are two independent transmuted Maxwell distributions then, then all four stochastic orderings exist.

Proof. Let $X \sim TM(\theta_1, \lambda_1)$ and $Y \sim TM(\theta_2, \lambda_2)$. The log-likelihood ratio of Y to X is

$$log\frac{f_Y(x)}{f_X(x)} = \frac{3}{2}log\left(\frac{\theta_1}{\theta_2}\right) + x^2\left(\frac{\theta_2-\theta_1}{\theta_1\theta_2}\right) + log\left[1-\lambda_2+\frac{4\lambda_2}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta_2}\right)\right] - log\left[1-\lambda_1+\frac{4\lambda_1}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta_1}\right)\right]$$

and the derivative of log-likelihood ratio with respect to x is

$$\frac{d}{dx}log\frac{f_Y(x)}{f_X(x)} = 2x\left(\frac{\theta_2-\theta_1}{\theta_1\theta_2}\right) - \frac{8\lambda_2}{\sqrt{\pi}}\frac{x^2}{\theta_2^{\frac{3}{2}}}\frac{\exp\left(\frac{-x^2}{\theta_2}\right)}{\left(1-\lambda_2+\frac{4\lambda_2}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta_2}\right)\right)} + \frac{8\lambda_1}{\sqrt{\pi}}\frac{x^2}{\theta_1^{\frac{3}{2}}}\frac{\exp\left(\frac{-x^2}{\theta_1}\right)}{\left(1-\lambda_1+\frac{4\lambda_1}{\sqrt{\pi}}\Gamma_{\frac{3}{2}}\left(\frac{x^2}{\theta_1}\right)\right)} \quad (13)$$

- Consider $\lambda_1 = \lambda_2 = \lambda$. The derivative of log-likelihood ratio with respect to x is negative for $\theta_1 > \theta_2$ and positive for $\theta_1 < \theta_2$. It means that $X \leq_{lr} Y$ for $\lambda_1 = \lambda_2 = \lambda$ and $\theta_2 > \theta_1$.
- When $\theta_1 = \theta_2 = \theta$. The derivative of log-likelihood ratio with respect to x is negative for $x > \lambda_2 > \lambda_1$ and positive for $x > \lambda_1 > \lambda_2$. It means that $X \leq_{lr} Y$ for $x > \lambda_1 > \lambda_2$ and $\theta_1 = \theta_2 = \theta$.

Therefour, we conclude that $X \leq_{lr} Y$ under parametanic space $\{\lambda_1 = \lambda_2 = \lambda \ and \ \ \theta_2 > \theta_1\}$ and $\{x > \lambda_1 > \lambda_2 \ \ and \ \ \theta_1 = \theta_2 = \theta\}$.Shaked and Shanthikumar

(1994) have shown that the following relation exists among four stochastic orderings of distributions listed earlier:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$
$$\Downarrow$$
$$X \leq_{st} Y$$

Hence using above expression and the result of likelihood ratio ordering, we can conclude that the likelihood ratio ordering, the usual stochastic, the hazard rate and the mean residual life ordering exist for the transmuted Maxwell distributions under the restricted parameter space.

4. Parameter Estimation

Let $x_1, x_2, \ldots, x_n$ (denoted by $\underline{x}$ henceforth) be a random sample of size n from TMW distribution with dendity function $f(t, \theta, \lambda)$ given in (5). Then the likelihood function of (θ, λ), in the light of given observations $\underline{x}$, comes out to be

$$L(\theta, \lambda; \underline{x}) = \left(\frac{4}{\sqrt{\pi}}\right)^n \theta^{\frac{-3n}{2}} \prod_{i=1}^{n} x_i^2 \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{\theta}\right) \prod_{i=1}^{n} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x_i^2}{\theta}\right)\right). \quad (14)$$

Log likelihood function is

$$logL(\theta, \lambda; \underline{x}) = nlog\left(\frac{4}{\sqrt{\pi}}\right) + \sum_{i=1}^{n} log(x_i^2) - \frac{3n}{2} log\theta - \frac{\sum_{i=1}^{n} x_i^2}{\theta} + \sum_{i=1}^{n} log\left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x_i^2}{\theta}\right)\right) \quad (15)$$

In order to obtain MLEs of parameters, we differentiate (15) w.r.t θ and λ and get likelihood equations. Since, the log-likelihood function contains incomplete gamma function, we use the formula of Gradshteyn & Ryzhik (1980) ($p. 18, \oint 0.410$), to derive likelihood equation for θ. The likelihood equation for θ is

$$\frac{-3n}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^2} + \sum_{i=1}^{n} \left(\frac{\psi'(x_i, \theta, \lambda)}{\psi(x_i, \theta, \lambda) + 1 - \lambda}\right) = 0, \quad (16)$$

where

$$\psi(\mathrm{y}, \theta, \lambda) = \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{\mathrm{y}^2}{\theta}\right)$$

and $\psi'(.)$ is the first derivative of $\psi(.)$ w.r.t. θ given by

$$\psi'(\mathrm{y}, \theta, \lambda) = \frac{4\lambda}{\theta^{\frac{5}{2}} \sqrt{\pi}} y^3 \exp\left(\frac{-\mathrm{y}^2}{\theta}\right).$$

Now, defferentiating (15) w.r.t. λ, we get the likelihood equation for λ given by

$$\sum_{i=1}^{n} \frac{\psi(x_i,\theta,\lambda)-\lambda}{\lambda(\psi(x_i,\theta,\lambda)+1-\lambda)} = 0. \quad (17)$$

The likelihood equations (16) and (17) are transcendental equations in θ and λ, respectively. We use iterative numerical method to solve these equations and obtain the MLEs of parameters.

4.1 Asymptotic Confidence Intervals

In previous section, we could not obtain expressions for *MLEs* of parameters in closed form and hence the exact distributions of the same can not be derived. Here, we derive ACIs for parameters by using the fact that the asymptotic distribution of *MLE* is normal distribution. Here, the Fisher information matrix is given by

$$I_{ij} = -\mathrm{E}\left(\frac{\partial^2 \log \mathrm{L}}{\partial\theta\,\partial\lambda}\right); \mathrm{i}, \mathrm{j} = 1, 2.$$

Since expressions for the elements of above matrix cannot be obtained in closed form, we therefore use approximate(observed) variance-covariance matrix given by

$$I(\hat{\theta}, \hat{\lambda}) = \begin{bmatrix} -\frac{\partial logL}{\partial\theta^2} & -\frac{\partial logL}{\partial\theta d\lambda} \\ -\frac{\partial logL}{\partial\lambda\,\partial\theta} & -\frac{\partial logL}{\partial\lambda^2} \end{bmatrix}^{-1}_{(\theta,\lambda)=(\hat{\theta},\hat{\lambda})} = \begin{bmatrix} \hat{\sigma}^2_\theta & \hat{\sigma}_{\theta,\lambda} \\ \hat{\sigma}_{\lambda,\theta} & \hat{\sigma}^2_\lambda \end{bmatrix}.$$

For our problem, the second derivatives of the log-likelihood function (15), are obtained as follows

$$\frac{\partial^2}{\partial\theta^2} logL = \frac{3n}{2\theta^2} - 2\frac{\sum_{i=1}^{n} x_i^2}{\theta^3} + \sum_{i=1}^{n}\left(\frac{\psi''(x_i,\theta,\lambda)}{\psi(x_i,\theta,\lambda)+1-\lambda} - \left(\frac{\psi'(x_i,\theta,\lambda)}{\psi(x_i,\theta,\lambda)+1-\lambda}\right)^2\right), \quad (18)$$

$$\frac{\partial^2}{\partial\lambda^2} logL = -\frac{1}{\lambda^2}\sum_{i=1}^{n}\left(\frac{\psi(x_i,\theta,\lambda)-\lambda}{\psi(x_i,\theta,\lambda)+1-\lambda}\right)^2, \quad (19)$$

and

$$\frac{\partial^2}{\partial\theta\,\partial\lambda} logL = \frac{\partial^2}{\partial\lambda\,\partial\theta} logL = \sum_{i=1}^{n} \frac{\psi'(x_i,\theta,\lambda)}{\lambda(\psi(x_i,\theta,\lambda)+1-\lambda)^2}, \quad (20)$$

where $\psi''(.)$ is the second derivative of $\psi(.)$ with respect to θ, given by

$$\psi''(\mathrm{y},\theta,\lambda) = \frac{2\lambda}{\theta^{\frac{9}{2}}\sqrt{\pi}}\mathrm{y}^3\exp\left(\frac{-\mathrm{y}^2}{\theta}\right)(2\mathrm{y}^2 - 5\theta). \quad (21)$$

Using the asypmtotic normality of MLE, two sided $100(1-\alpha)\%$ ACIs for θ and λ are given, respectively, by

$$\hat{\theta} \pm Z_{\left(\frac{\alpha}{2}\right)}\sqrt{\hat{\sigma}_{\theta}^{2}} \quad \text{and} \quad \hat{\lambda} \pm Z_{\left(\frac{\alpha}{2}\right)}\sqrt{\hat{\sigma}_{\lambda}^{2}}.$$

Where z_p represents the upper p^{th}percentile of standard normal distribution.

5 Bayesian estimation

Here, we suppose that θ is a random variables and the distribution of θ can be presented in form of a prior density $\pi_1(\theta)$. We consider the distribution of $\pi_1(\theta)$ to be Inverted gamma distribution with hyper parameters (ν, μ) given by the pdf

$$\pi_1(\theta) \propto \theta^{-(1+\nu)} e^{\frac{-\mu}{\theta}}, \quad \theta > 0$$

wheres for λ, we take a non-informative prior given by

$$\pi_2(\lambda) \propto \frac{1}{c}. \quad c > 0.$$

Considering θ and λ independent, the joint prior density comes out to be $\pi(\theta, \lambda) \propto \pi_1(\theta)\pi_2(\lambda)$. Now, merging the joint prior density with likelihood function via Bayes theorem, we get the hint posterior distribution of θ and λ given by

$$\pi\left(\theta, \lambda \middle| \underline{x}\right) \propto \theta^{-\left(1+\nu+\frac{3n}{2}\right)} exp\left(-\frac{1}{\theta}\left(\mu + \sum_{i=1}^{n} x_i^2\right)\right) \prod_{i=1}^{n} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x_i^2}{\theta}\right)\right). \quad (22)$$

The Bayes estimator of a parameter, under squared error loss function, is its posterior mean. Therefore, in order to obtain Bayes estimator of parameter, we need marginal densities of θ and λ. But, we observe that joint posterior density in (22) is not in a closed form and therefore, the marginal densities of θ and λ can not be expressed in closed form. We, therefore, use Gibbs Sampling method to evaluate Bayes estimates.

In order to implement Gibbs Sampler, the full conditional distribution of θ and λ are given by

$$\pi_1\left(\theta \middle| \lambda; \underline{x}\right) \propto \theta^{-\left(1+\nu+\frac{3n}{2}\right)} exp\left(-\frac{1}{\theta}\left(\mu + \sum_{i=1}^{n} x_i^2\right)\right) \prod_{i=1}^{n} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x_i^2}{\theta}\right)\right) \quad (23)$$

and

$$\pi_2\left(\lambda \middle| \theta; \underline{x}\right) \propto \prod_{i=1}^{n} \left(1 - \lambda + \frac{4\lambda}{\sqrt{\pi}} \Gamma_{\frac{3}{2}}\left(\frac{x_i^2}{\theta}\right)\right). \quad (24)$$

The full conditionals of θ and λ given in (23) and (24) do not follow any well known distribution. To generate the sample observations from then full conditionls, we use Metropolis-Hastings algorithm.

5.1 MCMC method

We use Metropolis-Hastings algorithm with normal proposal distribution to generate sample observation from $\pi_1(\theta|\underset{\sim}{x})$ and $\pi_2(\lambda|\underset{\sim}{x})$ given by equation(23) and (24) respectively. The main steps what are to be followed are as follows.

1. Set $t = 1$, and take θ_0=$\hat{\theta}$ and λ_0=$\hat{\lambda}$.
2. Generate a candidate point θ^* from proposal distribution q_1 ~N($\hat{\theta}$,v($\hat{\theta}$)) and generate a point u from uniform distribution $U(0,1)$.
3. Let $p(\theta_{(t-1)}, \theta^*) = min\left\{\frac{\pi_1(\theta^*|\lambda_{(t-1)};\underline{x})q_1(\theta_{(t-1)}|\theta^*)}{\pi_1(\theta_{(t-1)}|\lambda_{(t-1)};\underline{x})q_1(\theta^*|\theta_{(t-1)})}, 1\right\}$then set $\theta_{(t)} = \theta^*$ if $u \leq p(\theta_{(t-1)}, \theta^*)$ and otherwise set $\theta_{(t)} = \theta_{(t-1)}$.
4. Generate a candidate point λ^* from proposal distributionq_2~U$(-1, 1)$and generate a point u from uniform distribution $U(0,1)$.
5. Let $p(\lambda_{(t-1)}, \lambda^*) = min\left\{\frac{\pi_2(\lambda^*|\theta_{(t)},\underline{x})q_2(\lambda_{(t-1)}|\lambda^*)}{\pi_2(\lambda_{(t-1)}|\theta_{(t)},\underline{x})q_2(\lambda^*|\lambda_{(t-1)})}, 1\right\}$, then set $\lambda_{(t)} = \lambda^*$ if $u \leq p(\lambda_{(t-1)}, \lambda^*)$ and otherwise $\lambda_{(t)} = \lambda_{(t-1)}$.
6. set $t = t + 1$
7. Repeat steps $(2) - (6)$, K times to get the chains $\theta_1, \theta_2, \ldots, \theta_K$ and $\lambda_1, \lambda_2, \ldots, \lambda_K$, where K is a large number.

After the convergence of chain, we obtain K^* out of K observations, say $\theta_1, \theta_2, \ldots, \theta_{K^*}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{K^*}$ and taking average of these values. We can obtain Bayes estimate of θ and λ, under squared error loss function, by using

$$\tilde{\theta} = \frac{1}{K^*}\sum_{i=1}^{K^*} \theta_i \quad \text{and} \quad \tilde{\lambda} = \frac{1}{K^*}\sum_{i=1}^{K^*} \lambda_i.$$

5.2 Bayesian Credible interval

Here we previble procedure to obtain Bayesian credible intervals based on the algorithm of Chen and Shao (1999). The precedure includes the following steps:.

(i) Let $\theta_1 \leq \theta_2 \leq \theta_3 \leq, \ldots, \leq \theta_{K^*}$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq, \ldots, \leq \lambda_{K^*}$ be the ordered observations, obtained using Gibbs sampler, from the posterior distributions of θ and λ, respectively.

(ii) Then $100(1-\alpha)\%$ credible intervals for θ and λ are given by

$$\left(\theta_{\left[\left(\frac{\alpha}{2}\right)K^*\right]}, \theta_{\left[\left(1-\frac{\alpha}{2}\right)K^*\right]}\right) \text{ and } \left(\lambda_{\left[\left(\frac{\alpha}{2}\right)K^*\right]}, \lambda_{\left[\left(1-\frac{\alpha}{2}\right)K^*\right]}\right).$$

where [z] denotes the integer part of z.

5.3. Highest Posterior Density (HPD)

(i) Here, we first obtain length of $100(1-\alpha)\%$ credible interval for θ given by

$$l_j^{\theta} = \theta_{\left(j+\left(1-\frac{\alpha}{2}\right)K^*\right)} - \theta_{(j)}\ ; \quad j = 1, 2, 3, \ldots, \alpha K^*.$$

(ii) We find the credible interval for which l_j^{θ} is minimum. This interval is HPD for θ.

Similarly, we can obtain HPD for λ.

6. Simulation study

For simulation study, we generate the random samples from TMW by using acceptance-rejection method which is given as follows.

1. Generate a random variable Y from Rayleigh distribution.
2. Generate U from uniform distribution U(0, 1).
3. Set $X = Y$ (accept) If $U \leq \frac{f(Y)}{cg(Y)}$; where, c is $\sup_x \frac{f(Y)}{g(Y)}$,
   otherwise reject, and go back to step 1 .
4. repeate Steps 1-2, N times.

On the basis of these generated sample observations, we evaluate ML and Bayes estimates of parameters. For Bayesian estimation, the value of prior hyperparameters are chosen to be $\nu = \mu = 3$. We repeat this process 1000 times for different sample size and different values of parameters. Same procedure is followed in evaluation of length and coverage probabilities (CPs) of *ACIs*, *BCIs* and *HPD* intervals. Table 1 consists of average values of *ML* and Bayes estimates along with their mean square errors *(MSEs)*. In Table 2, we provide average length of *ACIs, BCIs* and *HPD* intervals along with *CP*. All computations are performed in R (https://www.r-project.org)software.

Table 1: Average value of ML and Bayes estimates with their MSE (in brackets()).

MLEs						
	θ =1.5,	λ=0.4	θ =2.0,	λ=0.6	θ =2.5,	λ=0.8
Sample Size	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
30	1.477 (0.126)	0.376 (0.099)	2.169 (0.131)	0.516 (0.096)	2.618 (0.177)	0.579 (0.083)
50	1.513 (0.081)	0.413 (0.055)	2.082 (0.078)	0.552 (0.036)	2.547 (0.124)	0.685 (0.025)

Bayes estimates						
	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\theta}$	$\tilde{\lambda}$
30	1.503 (0.001)	0.360 (0.071)	2.025 (0.003)	0.531 (0.081)	2.505 (0.002)	0.634 (0.098)
50	1.510 (0.001)	0.384 (0.046)	2.042 (0.003)	0.567 (0.027)	2.520 (0.001)	0.734 (0.009)

Table 2: Average length of ACIs, BCIs and HPD intervals with their coverage probability (in brackets ()).

	θ =1.5,	λ=0.4	θ =2.0,	λ=0.6	θ =2.5,	λ=0.8
Asymptotic confidence interval(ACI)						
Sample Size	θ	λ	θ	λ	θ	λ
30	1.480 (90.01)	1.720 (96.36)	1.174 (89.00)	1.115 (96.20)	2.200 (0.8062)	1.364 (90.91)
50	1.235 (93.70)	1.387 (97.70)	1.662 (86.30)	0.805 (98.70)	1.269 (87.00)	0.630 (93.20)
Bayesian credible interval(BCI)						
30	0.267 (96.59)	0.991 (96.0)	0.303 (99.60)	0.977 (90.20)	0.343 (95.22)	0.704 (88.30)
50	0.263 (95.44)	0.818 (94.34)	0.298 (98.70)	0.795 (96.80)	0.334 (95.62)	0.518 (92.40)
Highest posterior density(HPD) interval						
30	0.265 (94.31)	0.973 (93.80)	0.301 (99.80)	0.963 (92.10)	0.341 (92.99)	0.683 (94.60)
50	0.255 (95.44)	0.809 (94.21)	0.296 (98.90)	0.787 (98.40)	0.332 (94.67)	0.509 (97.80)

7. Real Data Analysis

Now we analyse a real data set which represents lifetimes of 23 ball bearing Lawless (2011).

17.88 28.92 33 41.52 42.12 45.60 48.4 51.84 51.96 54.12 55.56 67.80
68.64 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.4

To check whether the the distribution is suitable for this data set, K-S test is used. The K-S distance for this data is 0.1588 and p-value is 0.6073. The curves of empirical cdf and that of fitted cdf through TMW, are shown in Figure 3. Therefore TMW can be considered a suitable model for data. We also obtained MLEs, Bayes estimates, ACIs, BCIs and HPD intervals of parameters for this data set. For Bayesian estimation, the values of hypermeters are chosen to $\nu = \mu = 0$.

Table 3 : Estimated values for real data set

	θ	λ
ML	5389.99	0.498
Bayes	5219.99	0.3940
ACI	(2275.14, 8504.84)	(-0.4372, 1)
BCI	(4269.19, 6525.75)	(-0.4305, 0.9908)
HPD	(4233.68, 6471.03)	(-0.4288, 0.9782)

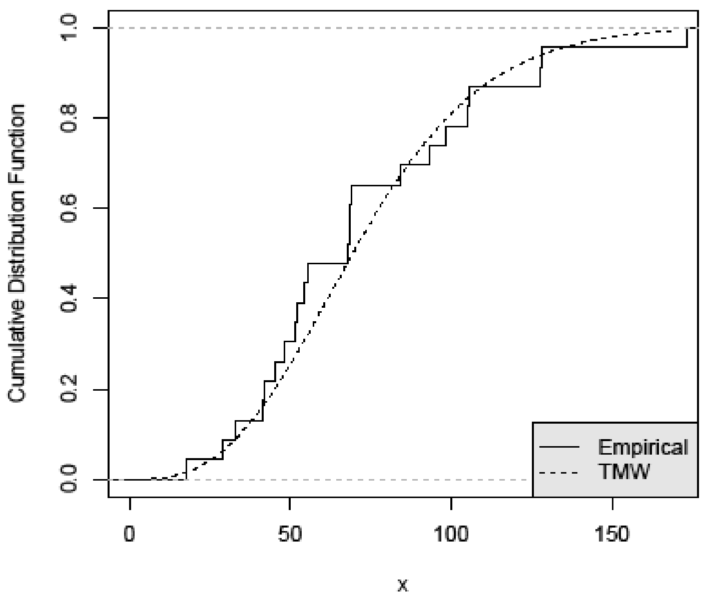

Fig. 3: Empirical and fitted cdfs

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